Tensor denoising and completion based on ordinal observations

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Ordinal tensor data in applications

- Tensor in networks (Human Connectome Project (HCP)).
- Each entry $y_\omega \in \{\text{high, moderate, low}\}$.

- Tensor in recommendation system (Baltrunas et al., 2011).
- Each entry $y_\omega \in \{1, 2, 3, 4, 5\}$

InCarMusic: Context-Aware Music Recommendations in a Car

4.1 Context Model and Music Track Corpus

In order to understand the influence of context on the music preferences of car passengers, context was modeled as a set of independent contextual factors. The factors are assumed to be independent in order to get a tractable mathematical model. This assumption, even if it is clearly false, as in other probabilistic models such as the naive Bayes classifier, still does not prevent the generation of reliable rating predictions. We identified the following factors and their possible values, contextual conditions:

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</tr>
<tr>
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Music tracks were of ten different genres. We observe that there is no unified music genre taxonomy, and we have chosen to use the genres defined in [12]: classical, country, disco, hip hop, jazz, rock, blues, reggae, pop and metal.

For phase one, i.e., the relevance assessment of different contextual factors, five representative tracks per genre were manually selected. This resulted in...
Challenges from ordinal tensor data

- Goal: learn a probabilistic tensor from multi-way ordinal observations.

1. The model should be invariant under a reversal of categories like \( \prec \rightarrow \text{neutral} \rightarrow \text{dislike} \) \( \iff \) \( \text{like} \rightarrow \text{neutral} \rightarrow \text{dislike} \).

2. The parameter interpretations should be consistent under merging or splitting of contiguous categories.

Two challenges for ordinal tensor model.

1. The entries do not belong to exponential family distribution.

2. The observation contains less information - neither the underlying signal nor the quantization operator is unknown.
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Summary of our contribution

- We establish the recovery theory for signal tensors and quantization operators simultaneously from observed ordinal tensor data.
- Let $\mathcal{Y} \in \mathbb{R}^{d \times \cdots \times d}$ be an order-$K$, $L$-level ordinal tensor.

<table>
<thead>
<tr>
<th></th>
<th>Bhaskar (2016)</th>
<th>Ghadermarzy et al. (2018)</th>
<th>This paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higher-order tensors ($K \geq 3$)</td>
<td>$\times$</td>
<td>$\checkmark$</td>
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</tr>
<tr>
<td>Multi-level categories ($L \geq 3$)</td>
<td>$\checkmark$</td>
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<td>$\checkmark$</td>
</tr>
<tr>
<td>Error rate for tensor denoising</td>
<td>$d^{-1}$ for $K = 2$</td>
<td>$d^{-(K-1)/2}$</td>
<td>$d^{-(K-1)}$</td>
</tr>
<tr>
<td>Optimality guarantee</td>
<td>unknown</td>
<td>$\times$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>Sample complexity for completion</td>
<td>$d^K$</td>
<td>$Kd$</td>
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- **Software:** [https://cran.r-project.org/web/packages/tensorordinal/index.html](https://cran.r-project.org/web/packages/tensorordinal/index.html)
Probabilistic model: a cumulative link model

- $[L] = \{1, 2, \cdots, L\}$ denotes the ordinal level.
- Let $\mathcal{Y} = [y_\omega] \in [L]^{d_1 \times \cdots \times d_K}$ be an ordinal tensor. The entries $y_\omega$ are independently distributed with cumulative probabilities:
  \[
P(y_\omega \leq \ell | b, \Theta) = f(b_\ell - \theta_\omega), \quad \text{for all } \ell \in [L - 1].
\]
- The additive, cumulative model enjoys two key properties for ordinal tensor data.
- If $f$ is a cumulative function,
  \[
P(y_\omega = \ell) = f(b_\ell - \theta_\omega) - f(b_{\ell - 1} - \theta_\omega) = P(b_{\ell - 1} < y^*_\omega \leq b_\ell),
\]
  where $\epsilon_\omega \stackrel{i.i.d.}{\sim} f$ and $y^*_\omega = \theta_\omega + \epsilon_\omega$.

ex) $f(x) = \frac{e^x}{1 + e^x}$ is a logistic link.
Latent variable interpretation

- We can interpret the ordinal tensor model (1) as an $L$-level quantization model.

![Diagram showing signal tensor, i.i.d. noise, and latent continuous-valued tensor](image)
Latent variable interpretation

- We can interpret the ordinal tensor model (1) as an $L$-level quantization model.

- Given intervals from the cut-off points vector $b$.

\[
\begin{align*}
\mathbf{y} & = \mathbf{\Theta} + \mathbf{\varepsilon} \\
\mathbf{y} & \equiv L\text{-level quantization} \\
\mathbf{y} & = \begin{cases} 
1, & \text{if } y_\omega^* \in (-\infty, b_1], \\
2, & \text{if } y_\omega^* \in (b_1, b_2], \\
\vdots & \vdots \\
L, & \text{if } y_\omega^* \in (b_{L-1}, \infty),
\end{cases}
\end{align*}
\]
The parameter $\Theta$ admits the Tucker decomposition:

$$\Theta = C \times_1 M_1 \times_2 \cdots \times_K M_K,$$

where $C \in \mathbb{R}^{r_1 \times \cdots \times r_K}$ is a core tensor, $M_k \in \mathbb{R}^{d_k \times r_k}$ are factor matrices.

Entries of $\Theta$ are uniformly bounded in magnitude by a constant $\alpha \in \mathbb{R}_+$. 
Rank constrained M-estimation

Let $\Omega \subset [d_1] \times \cdots \times [d_K]$ denote the set of observed indices. $\Omega$ could be the full set or incomplete set (for completion).

The log-likelihood associated with the observations is

$$
\mathcal{L}_{Y,\Omega}(\Theta, b) = \sum_{\omega \in \Omega} \sum_{\ell \in [L]} \left\{ \mathbb{1}_{y_{\omega}=\ell} \log [f(b_{\ell} - \theta_{\omega}) - f(b_{\ell-1} - \theta_{\omega})] \right\}.
$$

Here, $J = J_{1 \times \cdots \times K} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ denotes a tensor of all ones.
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  $\Omega$ could be the full set or incomplete set (for completion).
- The log-likelihood associated with the observations is
  $\mathcal{L}_{Y,\Omega}(\Theta, b) = \sum_{\omega \in \Omega} \sum_{\ell \in [L]} \left\{ 1_{\{y_\omega = \ell\}} \log \left[ f(b_\ell - \theta_\omega) - f(b_{\ell-1} - \theta_\omega) \right] \right\}$.
- We propose a rank-constrained maximum likelihood estimation for $\Theta$.

$$(\hat{\Theta}, \hat{b}) = \arg \max_{\Theta \in \mathcal{P}, b \in \mathcal{B}} \mathcal{L}_{Y,\Omega}(\Theta, b) \quad \text{where,}$$

$$\mathcal{P} = \{ \Theta \in \mathbb{R}^{d_1 \times \cdots \times d_K} : \text{rank}(\mathcal{P}) \leq r, \|\Theta\|_\infty \leq \alpha, \langle \Theta, J \rangle = 0 \}$$

$$\mathcal{B} = \{ b \in \mathbb{R}^{L-1} : \|b\|_\infty \leq \beta, \min_\ell (b_\ell - b_{\ell-1}) \geq \Delta > 0 \}.$$

Here, $J = [1] \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ denotes a tensor of all ones.
Theoretical results: tensor denoising

- **Tensor denoising:**
  - (Q1) How accurately can we estimate the latent signal tensor $\Theta$ from the ordinal observation $\mathcal{Y}$?
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  - (Q1) How accurately can we estimate the latent signal tensor $\Theta$ from the ordinal observation $\mathcal{Y}$?
  - (A1) Let us define $\text{MSE}(\hat{\Theta}, \Theta^{\text{true}}) = \prod_{k}^{1} \frac{1}{d_k} \|\hat{\Theta} - \Theta^{\text{true}}\|_F^2$.

---

**Statistical convergence (L. and Wang, 2020)**

With very high probability, our estimator $\hat{\Theta}$ satisfies

$$\text{MSE}(\hat{\Theta}, \Theta^{\text{true}}) \leq \min\left(4\alpha^2, c_1 r_{\max}^{K-1} \frac{\sum_k d_k}{\prod_k d_k}\right),$$

where $c_1 = c(f, K) > 0$ is a constant.
Theoretical results: tensor denoising

- **Tensor denoising:**
  - (Q1’) Is this bound optimal?
Theoretical results: tensor denoising

- **Tensor denoising:**
  - (Q1’) Is this bound optimal?
  - (A1’)

### Minimax lower bound (L. and Wang, 2020)

Under some mild technical conditions,

$$
\inf_{\hat{\Theta} \in \mathcal{P}} \sup_{\Theta^{\text{true}} \in \mathcal{P}} \mathbb{P}\left\{ \text{MSE}(\hat{\Theta}, \Theta^{\text{true}}) \geq c \min \left( \alpha^2, Cr_{\max} \frac{d_{\max}}{\prod_k d_k} \right) \right\} \geq \frac{1}{8},
$$

where $C = C(\alpha, L, f, b) > 0$ and $c > 0$ are constants independent of tensor dimension and the rank.

- So our estimation bound is rate-optimal.
Theoretical results: tensor completion

- **Tensor completion:**
  - (Q2) How many sampled entries do we need to consistently recover $\Theta$?

Let $\{y_{\omega}\}_{\omega \in \Omega}$ be the ordinal observation, where $\Omega$ is chosen at random with replacement according to a probability distribution $\Pi$. Then, with very high probability, $\|\Theta - \hat{\Theta}\|_F, \Pi \to 0$, as $|\Omega| \sum_k d_k \to \infty$.

The number of free parameters is roughly on the order of $\sum_k d_k$.

The sample complexity $|\Omega| \gg O(\sum_k d_k)$ is almost optimal.
Theoretical results: tensor completion

- **Tensor completion:**
  - (Q2) How many sampled entries do we need to consistently recover \( \Theta \)?
  - (A2) Let us define \( \| \Theta - \hat{\Theta} \|_{F, \Pi}^2 = \sum_{\omega \in [d_1] \times \cdots \times [d_K]} \pi_\omega (\Theta_\omega - \hat{\Theta}_\omega)^2 \).

### Sample complexity (L. and Wang, 2020)

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\[
\| \Theta - \hat{\Theta} \|_{F, \Pi}^2 \rightarrow 0, \quad \text{as} \quad \frac{|\Omega|}{\sum_k d_k} \rightarrow \infty.
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- The sample complexity \( |\Omega| \gg \mathcal{O}(\sum_k d_k) \) is almost optimal.
Algorithm

- The rank $r$ is unknown
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- The rank $r$ is unknown $\implies$ Bayesian information criterion (BIC).
- Non-convex problem $\implies$ Alternating optimization approach.
  - Let $\mathcal{L}_{\mathcal{Y},\Omega}(C, \mathcal{M}_1, \cdots, \mathcal{M}_K, b) = \mathcal{L}_{\mathcal{Y},\Omega}(\Theta, b)$.

**Algorithm:** Alternating optimization

**Result:** Estimated $\Theta$, together with core tensor and factor matrices

Random initialization;

**Repeat** until converge;

$C^{(n)} = \arg\max_C \mathcal{L}_{\mathcal{Y},\Omega}(C, \mathcal{M}_1^{(n-1)}, \cdots, \mathcal{M}_K^{(n-1)}, b^{(n-1)})$.

$\mathcal{M}_1^{(n)} = \arg\max_{\mathcal{M}_1} \mathcal{L}_{\mathcal{Y},\Omega}(C^{(n)}, \mathcal{M}_1^{(n)}, \cdots, \mathcal{M}_K^{(n-1)}, b^{(n-1)})$.

$\vdots$

$\mathcal{M}_K^{(n)} = \arg\max_{\mathcal{M}_K} \mathcal{L}_{\mathcal{Y},\Omega}(C^{(n)}, \mathcal{M}_1^{(n)}, \cdots, \mathcal{M}_k^{(n-1)}, b^{(n-1)})$.

$b^{(n)} = \arg\max_b \mathcal{L}_{\mathcal{Y},\Omega}(C^{(n)}, \mathcal{M}_1^{(n)}, \cdots, \mathcal{M}_K^{(n)}, b)$.

**end**
Simulations

- The decay in the error appears to behave on the order of $d^{-2}$ when $K = 3$.
- A larger estimation error is observed when the signal is too small or large.
- There is a big improvement from $L = 2$ to $L \geq 3$.

Figure: the relative MSE $= \| \hat{\Theta} - \Theta^{\text{true}} \|_F / \| \Theta^{\text{true}} \|_F$ for better visualization.
Data application: Human Connectome Project (HCP)

- An ordinal tensor consisting of structural connectivities among 68 brain nodes for 136 individuals.
- Each entry $y_\omega \in \{\text{high, moderate, low}\}$. 
Data application: Human Connectome Project (HCP)

- The clustering based on the estimated $\hat{\Theta}$ identifies 11 (3+8) clusters among 68 brain nodes.
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- The top three clusters capture the global separation among brain nodes.

- The small clusters represent local regions driving by similar nodes.
Data application: InCarMusic

- A tensor recording the ratings of 139 songs from 42 users on 26 contexts (Baltrunas et al., 2011).
- Each entry is a rating on a scale of 1 to 5 ($y_\omega \in \{1, 2, 3, 4, 5\}$).

![InCarMusic user interface (cont)](image)

(a) Tracks Proposed to Play  
(b) Rating a Track  
(c) Editing the User Profile  
(d) Configuring the Recommender

4.1 Context Model and Music Track Corpus

In order to understand the influence of context on the music preferences of car passengers, context was modeled as a set of independent contextual factors. The factors are assumed to be independent in order to get a tractable mathematical model. This assumption, even if it is clearly false, as in other probabilistic models such as the naive Bayes classifier, still does not prevent the generation of reliable rating predictions. We identified the following factors and their possible values, contextual conditions, a potentially relevant for InCarMusic recommendations:

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Data application: HCP, InCarMusic

- Our method achieves lower prediction error than others.

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<td>MAD</td>
<td>0.1607 (0.005)</td>
<td>0.2530 (0.0002)</td>
<td>0.3566 (0.0010)</td>
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<tr>
<td>MCR</td>
<td>0.1606 (0.005)</td>
<td>0.1599 (0.0002)</td>
<td>0.1563 (0.0010)</td>
</tr>
<tr>
<td>InCarMusic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAD</td>
<td>1.37 (0.039)</td>
<td>2.39 (0.152)</td>
<td>1.39 (0.003)</td>
</tr>
<tr>
<td>MCR</td>
<td>0.59 (0.009)</td>
<td>0.94 (0.027)</td>
<td>0.81 (0.005)</td>
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Table: Comparison of prediction error based on cross-validation (10 repetitions, 5 folds). Standard errors are reported in parentheses. MAD: mean absolute error; MCR: misclassification error.
Summary

- We propose a cumulative probabilistic model for ordinal tensor observations.
- The model achieves optimal convergence rate and nearly optimal sample complexity.
- The model has good interpretation and prediction performance in HCP and InCarMusic application.
- Thank you!


Software: https://cran.r-project.org/web/packages/tensorordinal/index.html